

Frobenius properties of tensor functors

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Abstract. This note is an announcement of my recent work on Frobenius properties of tensor functors between finite tensor categories. Fischman, Montgomery and Schneider showed that the Frobenius property of an extension A/B of finite-dimensional Hopf algebras is controlled by the modular functions of A and B . In this note, I explain how their result can be extended in the framework of finite tensor categories, a class of tensor categories including the representation category of a finite-dimensional Hopf algebra. I also introduce the “braided version” of their theorem.

1. Introduction

This note is an announcement of my recent work. An extension A/B of rings is said to be *Frobenius* if A is finitely-generated and projective as a right B -module and there is an isomorphism ${}_B A_A \cong \mathrm{Hom}_B(A_B, B_B)$ of B - A -bimodules. The Frobenius property of an extension of Hopf algebras has been studied in, *e.g.*, [6, 9, 10]. An important motivation for my work is the result of Fischman, Montgomery and Schneider [9] that says that the Frobenius property of an extension A/B of finite-dimensional Hopf algebras is controlled by the modular functions of A and B . In this note, I explain how their result can be extended to the framework of *finite tensor categories* [8], a class of tensor categories including the representation category of a finite-dimensional Hopf algebra.

2. A theorem of Fischman, Montgomery and Schneider

2.1. Basics on Hopf algebras

We first recall some basic results on Hopf algebras and fix related notations. By an *algebra* over a field k , we always mean an associative and unital algebra over k . A *Hopf algebra* over k is an algebra H endowed with algebra maps $\Delta : H \rightarrow H \otimes_k H$ and $\varepsilon : H \rightarrow k$ such that

$$\Delta(h_{(1)}) \otimes h_{(2)} = h_{(1)} \otimes \Delta(h_{(2)}), \quad \varepsilon(h_{(1)})h_{(2)} = h = h_{(1)}\varepsilon(h_{(2)}) \quad (2.1)$$

hold for all $h \in H$, and that there exists a linear map $S : H \rightarrow H$ satisfying

$$S(h_{(1)})h_{(2)} = \varepsilon(h)1_H = h_{(1)}S(h_{(2)}) \quad (2.2)$$

for all $h \in H$. Here, $h_{(1)} \otimes h_{(2)}$ is a symbolic notation for $\Delta(h) \in H \otimes_k H$. In view of (2.1), we will write

$$\Delta(h_{(1)}) \otimes h_{(2)} = h_{(1)} \otimes h_{(2)} \otimes h_{(3)} = h_{(1)} \otimes \Delta(h_{(2)})$$

for $h \in H$ (the Sweedler notation). The maps Δ and ε are called the *comultiplication* and the *counit* of H , respectively. The map S satisfying (2.2) is in fact unique. We call S the *antipode* of H .

Now let H be a Hopf algebra. A *right integral* in H is an element $\Lambda \in H$ such that $\Lambda h = \varepsilon(h)\Lambda$ for all $h \in H$. A *right cointegral* on H is a linear form λ on H such that $\lambda(h_{(1)})h_{(2)} = \lambda(h)1_H$ for all $h \in H$. It is known that the space of right (co)integrals is zero or one-dimensional. Moreover, a non-zero right integral in H exists if and only if H is finite-dimensional, and a non-zero right cointegral on H exists if and only if H is a co-Frobenius coalgebra; see, *e.g.*, [5].

From now on, we suppose that H is a finite-dimensional Hopf algebra. Then there exists a non-zero right integral $\Lambda \in H$. Since $h\Lambda$ ($h \in H$) is also a right integral, and since the space of right integrals is one-dimensional, we can define a map $\alpha_H : H \rightarrow k$ by $h\Lambda = \alpha_H(h)\Lambda$ ($h \in H$). The map α_H is an algebraic analogue of the modular function of a locally compact group, and therefore we also call α_H the *modular function* on H . It is easy to see that α_H is an algebra map. Thus α_H is also referred to as the *distinguished grouplike element* in H^* .

2.2. A theorem of Fischman, Montgomery and Schneider

Let A/B be an extension of finite-dimensional Hopf algebras over a field k , i.e., A is a finite-dimensional Hopf algebra and B is a Hopf subalgebra of A . We define the *relative modular function* $\chi_{A/B} : B \rightarrow k$ by

$$\chi_{A/B}(b) = \alpha_A(b_{(1)})\alpha_B(S(b_{(2)})) \quad (b \in B).$$

We also define the *relative Nakayama automorphism* $\beta_{A/B} : B \rightarrow B$ by

$$\beta_{A/B}(b) = \chi_{A/B}(b_{(1)})b_{(2)} = \alpha_A(b_{(1)})\alpha_B(S(b_{(2)}))b_{(3)} \quad (b \in B).$$

Write $\beta = \beta_{A/B}$ for simplicity. For a left B -module V , we denote by ${}_{\beta}V$ the left B -module obtained from V by twisting the action by β . The following result is an important motivation for my work:

Theorem 2.1 (Fischman-Montgomery-Schneider [9, Theorem 1.7]). *The extension A/B is a β -Frobenius extension in the sense that there exists an isomorphism ${}_B A_A \cong {}_{\beta}\mathrm{Hom}_B(A_B, B_B)$ of B - A -bimodules.*

Note that A is free as a right B -module (the Nichols-Zoeller theorem). Thus the above theorem implies that the extension A/B is Frobenius if $\beta = \mathrm{id}_B$. With a little more effort, we can see that the converse holds. In conclusion, the extension A/B is Frobenius if and only if $\beta = \mathrm{id}_B$, if and only if $\alpha_A|_B = \alpha_B$ [9, Corollary 1.8]

2.3. Categorical interpretation

Let A/B be an extension of finite-dimensional Hopf algebra. We consider the restriction functor $\mathrm{Res}_B^A : \mathrm{mod}\text{-}A \rightarrow \mathrm{mod}\text{-}B$, where $\mathrm{mod}\text{-}R$ means the category of finite-dimensional right R -modules. It is well-known that the functors $L = (-) \otimes_B A$ and $R = \mathrm{Hom}_B(A_B, -)$ are a left adjoint and a right adjoint of Res_B^A , respectively. The Nichols-Zoeller theorem implies that R is isomorphic to $(-) \otimes_B \mathrm{Hom}_B(A_B, B_B)$. Theorem 2.1 means that the “difference” between L and R is described by $\chi_{A/B}$.

The above argument is just a standard categorical interpretation of the notion of Frobenius-type properties of extensions of rings. There is a remarkable difference between our case and the case of an extension of ordinary rings. Namely, we can define the tensor product of modules over

a Hopf algebra by using the comultiplication, and the restriction functor Res_B^A preserves the tensor product of modules. We call such a functor a *tensor functor*. The following problem arises naturally:

Problem 2.2. Suppose that a tensor functor F has a left adjoint L and a right adjoint R . Describe the “difference” between L and R .

Theorem 2.1 is a complete answer to this problem with $F = \text{Res}_B^A$. In the next section, we give an answer to this problem in the case where F is a tensor functor between finite tensor categories such that L and R are exact.

3. Frobenius properties of tensor functors

3.1. Finite tensor categories

We first recall some categorical notions: First, a *monoidal category* is a category \mathcal{C} endowed with a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ (called the *tensor product*), an object $\mathbb{1} \in \mathcal{C}$ (called the *unit object*) and natural isomorphisms

$$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \quad \mathbb{1} \otimes X \cong X \cong X \otimes \mathbb{1} \quad (X, Y, Z \in \mathcal{C})$$

satisfying certain coherence conditions. If \mathcal{C} is a monoidal category, then \mathcal{C}^{op} is also a monoidal category. We write \mathcal{C}^{rev} to denote the category \mathcal{C} endowed with the reversed tensor product given by $X \otimes^{\text{rev}} Y = Y \otimes X$.

A *monoidal functor* is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between monoidal categories \mathcal{C} and \mathcal{D} endowed with a natural transformation $\xi_2 : F(X) \otimes F(Y) \rightarrow F(X \otimes Y)$ and a morphism $\xi_0 : \mathbb{1} \rightarrow F(\mathbb{1})$ in \mathcal{D} satisfying certain coherence conditions. We say that a monoidal functor (F, ξ_2, ξ_0) is *strong* if both ξ_2 and ξ_0 are invertible.

Let \mathcal{C} be a monoidal category. A *left dual object* of $X \in \mathcal{C}$ is an object $X^* \in \mathcal{C}$ endowed with morphisms $e : X^* \otimes X \rightarrow \mathbb{1}$ and $c : \mathbb{1} \rightarrow X \otimes X^*$ satisfying the so-called zig-zag relations. A left dual object is unique up to isomorphisms if it exists. If every object of \mathcal{C} has a left dual object, then \mathcal{C} is said to be *left rigid*. If this is the case, then the assignment $X \mapsto X^*$ extends to a strong monoidal functor $\mathcal{C}^{\text{op}} \rightarrow \mathcal{C}^{\text{rev}}$, called the *left duality*.

There are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X, Y \otimes Z) \cong \mathrm{Hom}_{\mathcal{C}}(Y^* \otimes X, Z), \quad (3.1)$$

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^*). \quad (3.2)$$

We say that \mathcal{C} is *right rigid* if $\mathcal{C}^{\mathrm{rev}}$ is left rigid. A *rigid monoidal category* is a monoidal category that is both left rigid and right rigid. If \mathcal{C} is rigid, then the contravariant functor $(-)^*$ is in fact an anti-equivalence on \mathcal{C} . We write $^*(-)$ to mean the inverse of $(-)^*$. Thus, there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X, {}^*Y \otimes Z) \cong \mathrm{Hom}_{\mathcal{C}}(Y \otimes X, Z), \quad (3.3)$$

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes {}^*Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y). \quad (3.4)$$

Definition 3.1 (Etingof-Ostrik [8]). A *finite tensor category* over k is a rigid monoidal category \mathcal{C} such that the following conditions are satisfied:

- (1) \mathcal{C} is a finite abelian category over k , i.e., \mathcal{C} is equivalent to $\mathrm{mod}\text{-}A$ for some finite-dimensional algebra A over k .
- (2) The tensor product \otimes of \mathcal{C} is k -linear in each variable.
- (3) $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong k$.

By (3.1)–(3.4), the tensor product of \mathcal{C} is exact in each variable.

3.2. The first theorem

By a *tensor functor*, we mean a k -linear exact strong monoidal functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between finite tensor categories. Note that a k -linear functor between finite abelian categories has a left (right) adjoint if and only if it is left (right) exact (a variant of the Eilenberg-Watts theorem). Now let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between finite tensor categories, and let L and R be a left adjoint and a right adjoint of F . Then we have:

Lemma 3.2. *The following assertions are equivalent:*

- (1) L is left exact.
- (2) R is right exact.

(3) $F(P)$ is projective whenever $P \in \mathcal{C}$ is projective.

Sketch of Proof. For a functor T between finite tensor categories, we set $T^! = {}^*T(-^*)$. It is easy to see that $S \dashv T$ (i.e., S is left adjoint to T) implies $T^! \dashv S^!$. Since a tensor functor preserves the duality, we have $R^! \dashv F^! \cong F$. Hence $R^! \cong L$. The equivalence (1) \Leftrightarrow (2) follows from this relation between L and R .

The implication (2) \Rightarrow (3) follows from $\mathrm{Hom}_{\mathcal{D}}(F(P), -) \cong \mathrm{Hom}_{\mathcal{C}}(P, -) \circ R$. To show the converse, we assume that $\mathcal{C} = \mathrm{mod}\text{-}A$ for some finite-dimensional algebra A . Then $F(A) \in \mathcal{C}$ is projective by the assumption. Thus $R \cong \mathrm{Hom}_A(A, R(-)) \cong \mathrm{Hom}_{\mathcal{D}}(F(A), -)$ is exact. \square

Now we state the first main result:

Theorem 3.3. *Let F , L and R be as above, and suppose that F satisfies the equivalent conditions of Lemma 3.2. Then there exists an object $\chi_F \in \mathcal{D}$ such that $L \cong R(\chi_F \otimes -)$. Such an object χ_F is unique up to isomorphism and invertible, i.e., $\chi_F \otimes \chi_F^* \cong \mathbb{1} \cong \chi_F^* \otimes \chi_F$. There are also isomorphisms*

$$L \cong R(- \otimes \chi_F), \quad L(\chi_F^* \otimes -) \cong R \cong L(- \otimes \chi_F^*).$$

Sketch of Proof. A \mathcal{C} -module category is a category \mathcal{M} endowed with a functor $\triangleright : \mathcal{C} \times \mathcal{M} \rightarrow \mathcal{M}$, called the *action*, and natural isomorphisms $(X \otimes Y) \triangleright M \cong X \triangleright (Y \triangleright M)$ and $\mathbb{1} \triangleright M \cong M$ satisfying certain axioms similar to those for a monoidal category. A \mathcal{C} -module functor is a functor between \mathcal{C} -module categories compatible with the actions. We use the fact that the class of \mathcal{C} -module functors is closed under taking adjoints.

By the assumption, R has a right adjoint, say G . The category \mathcal{C} is a \mathcal{C} -module category by the tensor product, and the category \mathcal{D} is a \mathcal{C} -module category by the action given by $X \triangleright V = F(X) \otimes V$ ($X \in \mathcal{C}, V \in \mathcal{D}$). Since F is a \mathcal{C} -module functor, R is so, and therefore G is so. Now we set $\chi_F = G(\mathbb{1})^*$. Then we have

$$G(X) = G(X \triangleright \mathbb{1}) \cong X \triangleright G(\mathbb{1}) = F(X) \otimes {}^*\chi_F$$

for all $X \in \mathcal{C}$. By definition, R is a left adjoint of G . On the other hand,

$$\mathrm{Hom}_{\mathcal{D}}(V, F(X) \otimes {}^*\chi_F) \cong \mathrm{Hom}_{\mathcal{D}}(V \otimes \chi_F, F(X)) \cong \mathrm{Hom}_{\mathcal{D}}(L(V \otimes \chi_F), X)$$

for $V \in \mathcal{D}$ and $X \in \mathcal{C}$. Hence $R \cong L(- \otimes \chi_F)$. We leave the rest of the proof. \square

Following this theorem, we introduce the following terminology:

Definition 3.4. We call the object χ_F the *relative modular object*.

After the workshop, I have learned two related works: Balan [1] studied the Frobenius-type property of Hopf monads and proved a similar result in more general setting. Balmer, Dell’Ambrogio and Sanders [2] showed such a result in the setting of tensor-triangulated categories. Thus, Theorem 3.3 may be an instance of a very general result in the category theory.

In any case, Theorem 3.3 is not sufficient as a generalization of the theorem of Fischman, Montgomery and Schneider. Their theorem describes the difference between L and R in terms of the modular functions, while Theorem 3.3 does not give any information about the object χ_F . Below, we give an explicit formula of χ_F in terms of a categorical analogue of the modular function.

3.3. The second theorem

Etingof, Nikshych and Ostrik [7] introduced the *distinguished invertible object* for a finite tensor category under the assumption that the base field k is algebraically closed. Their definition relies on the theory of exact module categories over \mathcal{C} . It is convenient to use the following alternative definition that requires less knowledge about the theory of finite tensor categories.

Definition 3.5 ([15]). Let $\text{REX}(\mathcal{C})$ denote the category of k -linear right exact endofunctors on \mathcal{C} . We define the *Cayley functor* by

$$\Upsilon_{\mathcal{C}} : \mathcal{C} \rightarrow \text{REX}(\mathcal{C}), \quad V \mapsto (-) \otimes V.$$

We also define $J_{\mathcal{C}} \in \text{REX}(\mathcal{C})$ by $J_{\mathcal{C}}(V) = \text{Hom}_{\mathcal{C}}(V, \mathbb{1})^* \cdot \mathbb{1}$ ($V \in \mathcal{C}$), where “ \cdot ” means the canonical action (often called the *copower*) of the category of finite-dimensional vector spaces on a finite abelian category. It can be shown that $\Upsilon_{\mathcal{C}}$ has a left adjoint. We let $\Upsilon_{\mathcal{C}}^*$ be a left adjoint of $\Upsilon_{\mathcal{C}}$ and define the *modular object* $\alpha_{\mathcal{C}} \in \mathcal{C}$ by

$$\alpha_{\mathcal{C}} = \Upsilon_{\mathcal{C}}^*(J_{\mathcal{C}}).$$

The finite tensor category \mathcal{C} is said to be *unimodular* if $\alpha_{\mathcal{C}} \cong \mathbb{1}$.

The distinguished invertible object $D \in \mathcal{C}$ of [7] is isomorphic to $\alpha_{\mathcal{C}}^*$ whenever D is defined. The modular object $\alpha_{\mathcal{C}}$ is invertible if the base field k is perfect, however, it is not known that whether it is invertible in general (this is why I used the different terminology to [7]). It is interesting to investigate what happens if k is an imperfect field.

If $\mathcal{C} = \text{mod-}H$ for some finite-dimensional Hopf algebra H , then $\text{REX}(\mathcal{C})$ can be identified with the category $H\text{-mod-}H$ of finite-dimensional H -bimodules. The Cayley functor corresponds to the composition

$$\text{mod-}H \xrightarrow{\sim} (\text{the category of Hopf } H\text{-bimodules}) \xrightarrow{\text{forget}} H\text{-mod-}H,$$

where the first arrow is the equivalence given by the fundamental theorem of Hopf bimodules. The functor $J_{\mathcal{C}}$ is isomorphic to $(-) \otimes_H k$. Using this observation, we see that the modular object $\alpha_{\mathcal{C}}$ is the one-dimensional H -module associated with the modular function α_H . In particular, $\alpha_{\mathcal{C}}$ is invertible in this case.

Now we state the second main result:

Theorem 3.6. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a tensor functor between finite tensor categories satisfying the equivalent conditions of Lemma 3.2. Then there exists an isomorphism $\chi_F \otimes \alpha_{\mathcal{D}} \cong F(\alpha_{\mathcal{C}})$.*

Thus, if $\alpha_{\mathcal{D}}$ is invertible, then $\chi_F \cong F(\alpha_{\mathcal{C}}) \otimes \alpha_{\mathcal{D}}^*$.

Sketch of Proof. As we have seen in the proof of Theorem 3.3, the functor R is a \mathcal{C} -module functor. In particular, $X \otimes R(V) \cong R(F(X) \otimes V)$ for $X \in \mathcal{C}$ and $V \in \mathcal{D}$. Using the Cayley functor, we can rewrite this as follows:

$$\text{REX}(F, R) \circ \Upsilon_{\mathcal{D}} \cong \Upsilon_{\mathcal{C}} \circ R,$$

where $\text{REX}(F, R)(T) = R \circ T \circ F$. Taking left adjoints, we get

$$\Upsilon_{\mathcal{D}}^* \circ \text{REX}(R, F) \cong F \circ \Upsilon_{\mathcal{C}}^*.$$

Evaluating both sides at $J_{\mathcal{D}} \in \text{REX}(\mathcal{D})$, we obtain $\chi_F \otimes \alpha_{\mathcal{D}} \cong F(\alpha_{\mathcal{C}})$. \square

We consider the case where $F = \text{Res}_B^A$ is the restriction functor associated with an extension A/B of finite-dimensional Hopf algebras. Then the relative modular object χ_F is the one-dimensional B -module associated with the relative modular function $\chi_{A/B}$, and the relative Nakayama automorphism $\beta_{A/B}$ corresponds to the functor $\chi_F \otimes (-)$. Hence we obtain Theorem 2.1. By the same argument, we easily get the “quasi-Hopf version” of Theorem 2.1.

4. Applications to braided Hopf algebras

4.1. Braided Hopf algebras

To obtain a meaningful result from our theorems, we need to know a description of the modular object in particular cases. There are few further results in this direction. In this section, we give a description of the modular object of the representation category of a Hopf algebra in a braided finite tensor category to establish the “braided version” of Theorem 2.1.

A monoidal category is said to be *braided* if it is endowed with a natural isomorphism $\sigma : \otimes \rightarrow \otimes^{\text{rev}}$ satisfying the so-called hexagon axiom. A *Hopf algebra* in a braided monoidal category \mathcal{V} (or a *braided Hopf algebra*) is an object $H \in \mathcal{V}$ endowed with structure morphisms

$$m : H \otimes H \rightarrow H, \quad u : \mathbb{1} \rightarrow H, \quad \Delta : H \rightarrow H \otimes H, \quad \varepsilon : H \rightarrow \mathbb{1}, \quad S : H \rightarrow H$$

satisfying the “braided version” of the axioms for an ordinary Hopf algebra. This notion reduces to an ordinary Hopf algebra in the case where \mathcal{V} is the category of vector spaces over k .

Now let \mathcal{V} be a braided finite tensor category. Given a Hopf algebra H in \mathcal{V} , we denote by \mathcal{V}_H the category of right H -modules in \mathcal{V} . It is easy to see that \mathcal{V}_H is a finite tensor category. Thus we can consider the modular object of \mathcal{V}_H . To describe it, we recall some results from the integral theory of braided Hopf algebras.

In the braided case, a right integral in H is a pair (X, f) consisting of an object $X \in \mathcal{V}$ and a morphism $f : X \rightarrow H$ satisfying $m \circ (f \otimes \text{id}_H) = \text{id}_X \otimes \varepsilon$ (under the canonical identification $H \otimes \mathbb{1} \cong H$). There is a right integral, denoted by $(\text{Int}(H), \Lambda)$, having a certain universal property. The

object $\text{Int}(H)$ is called the *object of integrals*. It is known that $\text{Int}(H)$ is invertible. Thus we can define the *right modular function* $\alpha_H : H \rightarrow \mathbb{1}$ by $\alpha_H \otimes \text{id}_{\text{Int}(H)} = m \circ (\text{id}_H \otimes \Lambda)$. See [4, 16] for details.

Theorem 4.1. *Let \mathcal{V} and H be as above. Then the modular object of $\mathcal{C} = \mathcal{V}_H$ is given as follows: As an object of \mathcal{V} , $\alpha_{\mathcal{C}} = \alpha_{\mathcal{V}} \otimes \text{Int}(H)^*$. The action is given by*

$$\alpha_{\mathcal{C}} \otimes H \xrightarrow{\text{id} \otimes \alpha_H} \alpha_{\mathcal{C}} \otimes \mathbb{1} \cong \alpha_{\mathcal{C}}.$$

The unimodularity of a finite tensor category is important in its application to topological invariants [14]. The following corollary is a direct consequence of the above theorem:

Corollary 4.2. *\mathcal{V}_H is unimodular if and only if $\alpha_H = \varepsilon$ and $\alpha_{\mathcal{V}} \cong \text{Int}(H)$.*

By an extension of Hopf algebras in \mathcal{V} , we mean a morphism $i_{A/B} : B \rightarrow A$ of Hopf algebras in \mathcal{V} that is monic as a morphism in \mathcal{V} . Theorems 3.3 and 3.6 yield the following “braided version” of Theorem 2.1:

Corollary 4.3. *Suppose that $\alpha_{\mathcal{V}}$ is invertible. For an extension $i_{A/B} : B \rightarrow A$ of Hopf algebras in \mathcal{V} , the following conditions are equivalent:*

- (1) *The restriction functor $\text{Res}_B^A : \mathcal{V}_A \rightarrow \mathcal{V}_B$ is Frobenius.*
- (2) *$\alpha_A \circ i_{A/B} = \alpha_B$ and $\text{Int}(A) \cong \text{Int}(B)$.*

4.2. Sketch of the proof of Theorem 4.1

The proof of Theorem 4.1 goes as follows: Let H be a Hopf algebra in a braided finite tensor category \mathcal{V} , and let $\mathcal{C} = \mathcal{V}_H$. We regard \mathcal{V} as a full subcategory of \mathcal{C} by regarding an object $V \in \mathcal{V}$ as a right H -module with the action $\text{id}_V \otimes \varepsilon$. There are obvious forgetful functors

$$\text{REX}_{\mathcal{C}}(\mathcal{C}) \xrightarrow{\Theta_{\mathcal{C}}} \text{REX}(\mathcal{C}), \quad \text{REX}_{\mathcal{C}}(\mathcal{C}) \xrightarrow{\Theta_{\mathcal{C}/\mathcal{V}}} \text{REX}_{\mathcal{V}}(\mathcal{C}) \xrightarrow{\Theta_{\mathcal{V}}} \text{REX}(\mathcal{C}),$$

where $\text{REX}_{\mathcal{C}}(\mathcal{C})$ is the category of k -linear right exact \mathcal{C} -module functors on \mathcal{C} and $\text{REX}_{\mathcal{V}}(\mathcal{C})$ is defined similarly. It turns out that Θ_{\square} ($\square = \mathcal{C}, \mathcal{V}, \mathcal{C}/\mathcal{V}$) has a left adjoint, say Θ_{\square}^* . Since $\Theta_{\mathcal{C}} = \Theta_{\mathcal{V}} \circ \Theta_{\mathcal{C}/\mathcal{V}}$, we have

$$\Theta_{\mathcal{C}}^* = \Theta_{\mathcal{C}/\mathcal{V}}^* \circ \Theta_{\mathcal{V}}^*. \quad (4.1)$$

For every $F \in \text{REX}_{\mathcal{C}}(\mathcal{C})$, we have $F(X) = F(X \triangleright \mathbb{1}) = X \otimes F(\mathbb{1})$ and thus F is determined by $F(\mathbb{1})$. This means that $\text{REX}_{\mathcal{C}}(\mathcal{C})$ can be identified with \mathcal{C} . Under this identification, the functor $\Theta_{\mathcal{C}}$ corresponds to the Cayley functor $\Upsilon_{\mathcal{C}}$. Hence, by the definition of the modular object, we have

$$\Theta_{\mathcal{C}}^*(J_{\mathcal{C}})(\mathbb{1}) \cong \Upsilon_{\mathcal{C}}^*(J_{\mathcal{C}}) = \alpha_{\mathcal{C}}. \quad (4.2)$$

Theorem 4.1 is obtained by computing $\Theta_{\mathcal{C}}^*(J_{\mathcal{C}})(\mathbb{1})$ by using the right-hand side of (4.1). For this purpose, we note that $\text{REX}_{\mathcal{V}}(\mathcal{C})$ is equivalent to the category ${}_H\mathcal{V}_H$ of H -bimodules in \mathcal{V} via ${}_H\mathcal{V}_H \rightarrow \text{REX}_{\mathcal{V}}(\mathcal{C}); M \mapsto (-) \otimes_H M$ (a variant of the Eilenberg-Watts theorem due to Pareigis [11–13]). In [15], the monad associated with $\Theta_{\mathcal{C}}^* \dashv \Theta_{\mathcal{C}}$ is described explicitly in terms of a certain algebra in $\mathcal{V} \boxtimes \mathcal{V}^{\text{rev}}$ used to define the modular object in [7]. Using this description, we have

$$\Theta_{\mathcal{V}}^*(J_{\mathcal{C}}) \cong (-) \otimes_H \alpha_{\mathcal{V}}, \quad (4.3)$$

where $\alpha_{\mathcal{V}}$ is regarded as an H -bimodule in \mathcal{V} by the counit of H .

The functor $\Theta_{\mathcal{C}/\mathcal{V}}^*$ can be described by the fundamental theorem for Hopf bimodules. Recall that a Hopf bimodule over H is an H -bimodule endowed with a left H -comodule structure compatible with the actions of H in a certain way. Let ${}^H_H\mathcal{V}_H$ be the category of Hopf bimodules over H . Bespalov and Drabant [3] showed that there is an equivalence $\mathcal{C} = \mathcal{V}_H \approx {}^H_H\mathcal{V}_H$ of categories (the fundamental theorem of Hopf bimodules).

Now let $F : {}^H_H\mathcal{V}_H \rightarrow {}_H\mathcal{V}_H$ be the functor forgetting the left H -comodule structure. Then the composition

$$\text{REX}_{\mathcal{C}}(\mathcal{C}) \xrightarrow{\approx} \mathcal{V}_H \xrightarrow{\approx} {}^H_H\mathcal{V}_H \xrightarrow{F} {}_H\mathcal{V}_H \xrightarrow{\approx} \text{REX}_{\mathcal{V}}(\mathcal{C})$$

is isomorphic to $\Theta_{\mathcal{C}/\mathcal{V}}$. Since a left adjoint of F is given by tensoring the left H -comodule H^* , the functor $\Theta_{\mathcal{C}/\mathcal{V}}^*$ is given by the composition

$$\text{REX}_{\mathcal{V}}(\mathcal{C}) \xrightarrow{\approx} {}_H\mathcal{V}_H \xrightarrow{H^* \otimes (-)} {}^H_H\mathcal{V}_H \xrightarrow{\approx} \mathcal{V}_H \xrightarrow{\approx} \text{REX}_{\mathcal{C}}(\mathcal{C}).$$

Finally, we use the integral theory for braided Hopf algebras to express the Hopf bimodule H^* in terms of $\text{Int}(H)$ and α_H . The proof of Theorem 4.1 is completed by combining the result with (4.1)–(4.3).

5. Summary and concluding remarks

Fischman, Montgomery and Schneider [9] showed that the Frobenius property of an extension A/B of finite-dimensional Hopf algebras is controlled by the modular functions of A and B . I generalized their result to tensor functors between finite tensor categories: The Frobenius property of such a functor is controlled by the modular objects (Theorems 3.3 and 3.6). I also give a description of the modular object of the representation category of a Hopf algebra in a braided finite tensor categories (Theorem 4.1). As an application, the “braided version” of their theorem is obtained (Corollary 4.3).

There are many results on finite-dimensional Hopf algebras involving the modular functions, and some of them have been generalized to the setting of finite tensor categories; see, *e.g.*, [7, 14]. Mentioning these results, I believe that the modular object is an important subject in the theory of finite tensor categories and needs further study (*e.g.*, the case over an imperfect field).

I also remark that Fischman, Montgomery and Schneider studied not only an extension of finite-dimensional Hopf algebras but also an extension of more general objects such as universal enveloping algebras of Lie color algebras. Technically, my approach depends on the finiteness of the categories and does not cover any results in the infinite-dimensional cases. I will try to remove the finiteness assumption in future work to understand several results on infinite-dimensional Hopf algebras from the category-theoretical point of view.

Acknowledgments

I would like to thank the organizers of Tsukuba Workshop on Infinite-dimensional Lie Theory and Related Topics. I am supported by Grant-in-Aid for JSPS Fellows (24-3606).

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(Received March 26, 2015)